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On the Reflection of Waves from a Stratum of Gradually Varying Properties, with Application to Sound

J. W. Nicholson

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is equivalent to about 0·3 of a cub. mm. per day. Considering I have found a rate of helium production of the order of 0·37 cub. mm., the agreement between experiment and the theoretical prophecy of Rutherford is almost too wonderful, substantiating as it does the accuracy of the theory of radio-active changes he has done so much to initiate and develop.

I have to express my obligations to Mr. Robert Lennox, F.C.S., and Mr. W. J. Green, B.Sc., for aid given in the conduct of these long and laborious experiments.

On the Reflection of Waves from a Stratum of Gradually Varying Properties, with Application to Sound.

By J. W. Nicholson, D.Sc., B.A., Isaac Newton Student, Scholar of Trinity College, Cambridge.

(Communicated by Professor J. Larmor, Sec. R.S. Received June 11,—Read June 25, 1908.)

In a variable medium, the velocity of propagation of a train of waves, and the wave-length at any point, are functions of the position of that point. The circumstances of such a propagation have only been worked out in detail in one particular case. Lord Rayleigh,[*] in connection with the transverse vibrations of a string of variable density, dealt very completely with the case in which the density is inversely proportional to the distance from a fixed point. In his original investigation[†] the results were applied to the corresponding optical problem, and a numerical example given.

Although this is perhaps the only interesting case in which a simple exact solution appears possible, yet a close approximation may be made to the existing conditions, even in the general problem, when the waves are short in comparison with the other distances concerned. The development of such a theory, with an examination of some important cases, is the object of the present paper.

Let V_0 be the velocity of a plane wave-train at some point of a medium, which we may choose as origin, and suppose that the train is advancing along the direction x.

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[*] 'Theory of Sound,' vol. 1, § 148.

[†] 'Proc. Lond. Math. Soc.,' vol. 11, 1880, pp. 51—56; 'Collected Papers,' vol. 1, pp. 460—465.

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The velocity V at the point defined by x is taken as a function of x only. In order that the results may not be restricted to any particular class of vibrations, let ϕ be the vector, of whatever character, whose propagation defines the vibration.

In all important cases, ϕ may be chosen such a vector that the surface conditions are

(i) ϕ is continuous,
(ii) $d\phi/dx$ is continuous.

The equation of propagation $\dfrac{d^2\phi}{dt^2} = \mathrm{V}^2\dfrac{d^2\phi}{dx^2}$ yields for a simple wave-train

$$d^2\phi/dx^2+k^2\phi = 0, \tag{1}$$

where $2\pi/k$ is the wave-length λ, which is a function of x.

If k_0 is the value of k at the origin, we may assume a relation

$$k = k_0 f(x). \tag{2}$$

When k_0 is preponderant over $f(x)$, the approximate solution of the differential equation may be obtained by a method described by H. A. Webb,* after Horn, but employed earlier as a working method by Stokes† and L. Lorenz.‡

Write
$$\phi = e^{\pm \iota k\omega}\psi, \tag{3}$$

where ω and ψ are functions of x to be suitably chosen. If accents denote differentiations with respect to x, we obtain

$$\psi''\pm\iota k_0(\psi\omega''+2\psi'\omega')-k_0^2\psi(\omega'^2-f^2(x)) = 0.$$

Equating the coefficients of k_0 and k_0^2 separately to zero, since the terms are of different orders of magnitude, we obtain, if A is an arbitrary constant, and if the first term proves negligible, as will appear in the cases treated,

$$\omega = \int^x \sqrt{f(x)}\,dx, \qquad \psi = \mathrm{A}/\sqrt{\omega'}. \tag{4}$$

Thus the general solution, when the variable part of $k_0\omega$ is small compared with k_0, of the equation

$$\partial^2\phi/\partial x^2+k_0^2\omega'^2\phi = 0, \tag{5}$$

where ω' represents $\partial\omega/\partial x$, is obtained in the form

$$\phi = (\mathrm{A}e^{\iota k_0\omega}+\mathrm{B}e^{-\iota k_0\omega})/\omega'^{\frac{1}{2}}. \tag{6}$$

We proceed to discuss the reflection backward of waves travelling along x, in traversing a medium of this slowly changing character, and of extent large compared with the wave-length. Let the medium be uniform from

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* 'Roy. Soc. Proc.,' vol. 74, 1904, p. 315.

† 'Math. and Phys. Papers,' vol. 2, p. 334.

‡ 'Œuvres Scientifiques,' vol. 1, p. 435 *et seq.*

$x = -\infty$ to $x = x_1$, slowly variable from $x = x_1$ to $x = x_2$, and again uniform, but differently so, from $x = x_2$ to $x = \infty$, there being perfect material continuity at x_1 and x_2, and the values of ω in the extreme media being ω_1 and ω_2.

A wave advancing along x positive into the varying stratum is of type

$$\phi_1 = \mathrm{H}e^{-\iota k_0 \omega_1' (x-x_1)}, \tag{7}$$

to which corresponds a reflected wave, emerging from it,

$$\phi_2 = \mathrm{K}e^{\iota k_0 \omega_1' (x-x_1)}. \tag{8}$$

The time factor, say $e^{\iota ct}$, is included in H and K.

In the intermediate variable stratum, the complete direct and reflected disturbance is given by (6), and in the final uniform medium, where there can be no negative wave,

$$\phi = \mathrm{C}e^{-\iota k_0 \omega_2' (x-x_1)}. \tag{9}$$

If μ_{12} is the refractive index between the terminal media in the optical case, then $\mu_{12} = \omega_2'/\omega_1'$.

The surface conditions at $x = x_0$, $x = x_1$ lead to

$$(\mathrm{H}+\mathrm{K})\,\omega_1'^{\frac{1}{2}} = \mathrm{A}e^{\iota k_0 \omega_1} + \mathrm{B}e^{-\iota k_0 \omega_1},$$

$$\mathrm{C}\omega'^{\frac{1}{2}} = \mathrm{A}e^{\iota k_0 \omega_2} + \mathrm{B}e^{-\iota k_0 \omega_2}$$

$$-\iota k_0\,(\mathrm{H}-\mathrm{K})\,\omega_1'^{\frac{1}{2}} = \mathrm{A}e^{\iota k_0 \omega_1}(\iota k_0 - \omega_1''/2\omega_1'^2) - \mathrm{B}e^{-\iota k_0 \omega_1}(\iota k_0 + \omega_1''/2\omega_1'^2),$$

$$-\iota k_0 \mathrm{C}\omega_2'^{\frac{1}{2}} = \mathrm{A}e^{\iota k_0 \omega_2}(\iota k_0 - \omega_2''/2\omega_2'^2) - \mathrm{B}e^{-\iota k_0 \omega_2}(\iota k_0 + \omega_2''/2\omega_2'^2).$$

If $s = e^{\iota k_0(\omega_2 - \omega_1)}$, $\quad 2\iota k_0\,(\epsilon_1, \epsilon_2) = (\omega_1''/\omega_1'^2,\ \omega_2''/\omega_2'^2)$,

where (ϵ_1, ϵ_2) are small pure imaginaries of the order of a wave-length, then

$$\mathrm{H}\begin{vmatrix} 2-\epsilon_1 & -\epsilon_1 & 0 \\ s & s^{-1} & 1 \\ s(1-\epsilon_2) & -s^{-1}(1+\epsilon_2) & -1 \end{vmatrix} = \mathrm{K}\begin{vmatrix} \epsilon_1 & 2+\epsilon_1 & 0 \\ s & s^{-1} & 1 \\ s(1-\epsilon_2) & -s^{-1}(1+\epsilon_2) & -1 \end{vmatrix}$$

Thus $$\mathrm{K}/\mathrm{H} = \{\epsilon_2(2-\epsilon_1) - s^2\epsilon_1(2-\epsilon_2)\}/\{\epsilon_1\epsilon_2 + s^2(2+\epsilon_1)(2-\epsilon_2)\}, \tag{10}$$

which determines the amplitude and phase of the reflected wave.

At present we consider only the case in which ω_1 and ω_2 are real. The square of the wave-length being small compared with that of the distance traversed, as required for this analysis, the result takes a simpler form.

Write $$\omega_2 - \omega_1 = \beta, \qquad 2k_0\,(\epsilon_1, \epsilon_2) = \iota\,(\alpha_1, \alpha_2), \tag{11}$$

so that $$\alpha = -\omega''/\omega'^2,$$

and is purely real. Then

$$\mathrm{K}/\mathrm{H} = (\epsilon_2 - \epsilon_1 \exp 2\iota k_0\beta)/2 \exp 2\iota k_0\beta$$
$$= (\alpha_2 \sin 2k_0\beta - \iota\alpha_1 + \iota\alpha_2 \cos 2k_0\beta)/4k_0.$$

If R is the ratio of the intensities of the reflected and incident waves,

$$\mathrm{R} = (\alpha_1^2 + \alpha_2^2 - 2\alpha_1\alpha_2 \cos 2k_0\beta)/16k_0^2; \tag{12}$$

also the increase of phase in the reflection, measured at the front x_1, is θ, where

$$\tan\theta = (\alpha_2 \cos 2k_0\beta - \alpha_1)/\alpha_2 \sin 2k_0\beta. \tag{13}$$

Higher approximations may be found if desired; the value of the function ψ previously used may be found to a higher order by writing, in the equation of propagation,

$$\phi = e^{\iota k_0 \omega}\left(\psi_1 + \frac{\psi_2}{k_0} + \frac{\psi_3}{k_0{}^2} + \dots\right). \tag{14}$$

Verification of the Approximation.

The result may be tested by comparison with Lord Rayleigh's solution for a particular case previously referred to, which is mathematically exact.

In that case $\omega' = x^{-1}, \qquad \alpha = -\omega''/\omega'^2 = 1,$

$$\beta = \omega_2 - \omega_1 = \log . x_2/x_1 = \log \mu,$$

where μ is the refractive index between the extreme media, which is to be not much different from unity.

Thus $$\mathrm{I} = \sin^2(k_0 \log \mu)/4k_0{}^2, \tag{15}$$

while Lord Rayleigh's result is

$$\mathrm{I} = \sin^2(k_0 \log \mu)/\{4k_0{}^2 + \sin^2(k_0 \log \mu)\},$$

so that the agreement is close even for only moderately large values of k_0.

Effect of an Absorbing Medium.

If the second medium have an absorbent effect, the differential equation relating to it has a term proportional to the velocity, and becomes

$$\partial^2\phi/\partial t^2 + \sigma\partial\phi/\partial t = \mathrm{V}^2\partial^2\phi/\partial x^2, \tag{16}$$

where V is the velocity of propagation, and σ is a certain constant whose value must be positive. A negative value would denote instability of the medium.*

If ϕ has a time factor $e^{\iota ct}$, and $2\pi/k$ is the wave-length at the origin as before,

$$\partial^2\phi/\partial x^2 + (k^2 - \iota \mathrm{C}\sigma)\,\phi = 0, \tag{17}$$

and absorption may be accounted for by taking the function ω as complex. But the imaginary portion must be essentially negative.

If, therefore, the second medium be absorbent, ϵ_2, and therefore β, is complex, and we may write

$$\beta = \lambda - \iota\mu,$$

where λ and μ are positive.

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* *Cf.* a discussion by Lord Rayleigh, of Cauchy's "Theory of Reflection by Opaque Bodies," 'Phil. Mag.,' vol. 42, 1871, p. 441; 'Collected Works,' vol. 1, p. 145.

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Thus, in previous notation,

$$\begin{aligned}\mathrm{K}/\mathrm{H} &= (\epsilon_2 - \epsilon_1 \exp 2\iota k_0\beta)/2 \exp 2\iota k_0\beta \\ &= (\epsilon_2 - \epsilon_1 \exp 2k_0\mu \exp 2\iota k_0\lambda)/2 \exp 2k_0\mu \exp 2\iota k_0\lambda.\end{aligned}$$

Even when μ is fairly small the function $\exp 2k_0\mu$ may be very large, if the wave-length be small enough, and approximately, in this case,

$$\mathrm{K}/\mathrm{H} = -\tfrac{1}{2}\epsilon_1 = \iota\alpha_1/4k_0.$$

The ratio of the intensities of the reflected and incident waves becomes

$$\mathrm{R} = \alpha_1{}^2/16\mathrm{K}_0{}^2. \tag{18}$$

A very moderate amount of absorption is sufficient, in these circumstances, to render the reflected wave practically independent of it, and to destroy entirely the periodic effects with changing thickness of layer, which obtain in the absence of absorption. The value of R in (18) may be called the "limiting" intensity.

In order to obtain a general idea of the validity of this reasoning, it is sufficient to consider the special case of linear variation of the medium, which will usually occur in practice. The amount of variation in the medium, of which this analysis can take account, must first be estimated.

In obtaining (4), ψ'' was neglected, where ψ then became $\mathrm{A}/\sqrt{\omega'}$.

The ratio borne to ψ'' by the greatest term retained in the differential equation is of order

$$k_0\psi\omega''/\psi'' = k_0\omega''\omega^{-\frac{1}{2}}/(\tfrac{3}{4}\omega''/\omega'^{\frac{5}{2}} + \ldots)\,;$$

and for a law of variation given by

$$\omega' = 1 + cx,$$

where x is distance from the origin 0, this becomes

$$4k_0\omega'(1+cx)/3c, \quad \text{or } 8/c\lambda$$

practically, on reduction, where λ is the "mean" wave-length.

Thus if c is so great that $c\lambda = \frac{1}{2}$, the error involved is not greater than about 6 per cent. This may be taken as an extreme case in which the analysis is approximately correct.

If c is entirely due to absorption, it may be written $-\iota\mu$, where μ is real and positive, and $2\mu\lambda$ may be as great as unity. Thus $\omega' = 1 - \iota\mu x$,

$$[\omega]_1{}^2 = [x - \tfrac{1}{2}\iota\mu x^2]_1{}^2 = \beta,$$

and if d is the thickness of the stratum, $\exp 2\iota k_0\beta$ contains a factor $\exp \mu k_0 d^2$, whose value is approximately $\exp (3d^2/\lambda^2)$ in the extreme case $2c\lambda = 1$, where λ is again the "mean" wave-length. This quantity is very large even if d is but a wave-length. The result obtained for the general case is therefore justified under these circumstances.

The formula for the transmitted wave discussed below indicates that in a case like the present the intensity of that wave contains a factor $\exp(-2\iota k_0\beta)$, or $\exp(-3d^2/\lambda^2)$ when the absorption is extreme. It therefore appears that the disappearance of the periodic effect with varying thickness of layer corresponds, in general, to an absorption which is very complete in less than a wave-length.

The analysis has been shown capable of taking account of this small thickness of stratum, provided the rate of change be not too rapid. For any wave-length there is a limiting amount of change to which the method applies, given by $2c\lambda = 1$ within an error of 6 per cent. A greater absorption requires a smaller mean wave-length.

The Transmitted Wave.

In the more general case with previous notation, it may be readily shown that

$$C/H = 4s(\omega_1'/\omega_2')^{\frac{1}{2}}/\{\epsilon_1\epsilon_2+s^2(2+\epsilon_1)(2-\epsilon_2)\}, \tag{19}$$

which determines the amplitude and phase of the transmitted wave.

When ϵ_2 is real, we may neglect ϵ_1, ϵ_2, and also $\epsilon_1\epsilon_2$ in comparison with unity.

Thus $$C/H = 1/s\sqrt{\mu_{12}},$$

and the incident wave, $\phi = H\cos\{k_0\omega_1'(x-x_1)-Vt\}$, leads to a transmitted undulation,

$$\phi_2 = H\mu_{12}^{-\frac{1}{2}}\cos\{k_0\omega_2'(x-x_1)+k_0(\omega_2-\omega_1)-Vt\}. \tag{20}$$

When β is complex of the form $\lambda-\iota\mu$, and μ is positive, s contains a factor, $\exp -k_0\mu$, making it very small. Since this expression in the numerator of (19) will completely outweigh the smallness of the denominator, there is no appreciable transmitted wave. It is, in fact, wholly absorbed, in accordance with the character then possessed by the second medium.

Discussion of the Results.

Reverting to the case in which ω_2 is real, to which corresponds the formula (12), we note that the intensity of reflection, for increasing thickness of the layer, has maxima and minima after the manner of Newton's rings in optics. But since $\alpha_1^2+\alpha_2^2-2\alpha_1\alpha_2$ is essentially positive, absolute extinction of the reflected waves is only possible when this quantity is zero, and therefore when $\alpha_1 = \alpha_2$.

The possible laws of slow variation to enable this to happen apart from those of periodic character are contained in ω''/ω'^2 constant or

$$\omega' = a/(bx+c), \tag{21}$$

which leads only to the case discussed by Lord Rayleigh. For all other laws, maxima and minima of reflection occur as the thickness of the layer varies.

The change of phase in (13) is measured at the plane $x = x_1$, where the waves enter the second medium. If the phase is unaltered, except by reversal,

$$\alpha_2 \cos 2k_0\beta = \alpha_1,$$

whence

$$\cos 2k_0\beta = \mu_2^2\left(\frac{d\mu}{dx}\right)_1 \bigg/ \mu_1^2\left(\frac{d\mu}{dx}\right)_2,$$

on putting $\omega' = \mathrm{C}\mu$, where C is constant and μ is the index of refraction at a point. This equation can be satisfied by real values if

$$\frac{d}{dx_2}\left(\frac{1}{\mu_2}\right) \text{ does not exceed } \frac{d}{dx_1}\left(\frac{1}{\mu_1}\right). \tag{22}$$

When limiting reflection is attained, owing to complete but very gradual absorption in the second slightly different medium, the change of phase must be $\frac{1}{2}\pi$. This is a case in which absorption takes place almost entirely in a wave-length.

Reflection of Sound from a Fog.

The results have some bearing upon the question of the action of a fog upon incident sound waves. The observations of Tyndall* and Henry† showed that sounds of different pitches vary their order of effectiveness, at a distance, in a remarkable manner. Henry made certain conjectures as to the motion of the air overhead, and regarded the consequent refraction‡ as the agent causing the peculiarities shown by distant sounds. Tyndall, however, whose view seems to be in accordance with many of the observations on fog-signals, postulated the existence of a flocculent condition of the atmosphere in such cases, caused by unequal heating, or by the presence of an excess of moisture in certain parts. The views of Stokes and Henry have been developed independently by Osborne Reynolds,§ who points out that, since wind raises sound by causing it to move faster below than above, any other cause producing such a difference in velocity of propagation must also lift the sound. Such a cause is the temperature gradient in the atmosphere. Adopting the results of Glaisher's balloon observations,|| Reynolds has calculated that the range of Tyndall's sounds might at any time have been extended by a quarter of a mile, had he been able to ascend 30 feet.

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* 'Phil. Trans.,' vol. 164 1874, p. 183.

† 'Report of the U.S. Naval Board for 1874.'

‡ Stokes, 'Math. and Phys. Papers,' vol. 4, p. 110.

§ 'Roy. Soc. Proc.,' vol. 22, 1874, p. 296.

|| 'Brit. Ass. Report,' 1862, p. 462.

Although Reynolds has pointed out that some of Tyndall's own observations may be explained by refraction, yet Lord Rayleigh* has cited others which appear explicable only by acoustic clouds. Moreover, the explanation by refraction of the increase in the range of sound when the sky becomes cloudy, or when evening approaches, appears to admit of an alternative. An atmosphere with much aqueous vapour present has a greatly increased power of radiating, as well as of absorbing heat, and the removal of the sun's effect might therefore produce the acoustic phenomenon in question.

The formulæ given above allow an estimate of the efficiency of the cause assigned by Tyndall. A portion of the numerator of the expression (12) obtained for the reflected intensity is periodic, and depends both upon the period of the sound and the degree of heterogeneity of the medium. For the present we are neglecting the small radiation of heat which takes place during the passage of sound. A slight change in k_0 does not appreciably modify the denominator of (12), but can cause a great alteration in the numerator: and sounds whose periods are not far apart can thus exhibit, under certain circumstances, a fair amount of difference in effectiveness. This is in accordance with observations. Since, moreover, k_0 is large compared with the varying part of β, a fairly small amount of variation of the medium, not greatly changing α_1 and α_2, may cause a more considerable change in $k_0\beta$ which may suffice to change the order of effectiveness of two sounds. A moderately small change in β may cause $\sin 2k_0\beta$ to oscillate between the limits ± 1, so that the corresponding intensity of reflected sound oscillates between the limits

$$(\alpha_1 \pm \alpha_2)^2/16k_0^2, \tag{23}$$

which may differ fairly widely.

This line of reasoning seems capable of explaining many of the vagaries of sounds coming from a distance. In foggy weather the atmosphere is more homogeneous than usual, and the backward reflection is extremely small. The sounds then have little tendency to vary their order of effectiveness, and signals are also effective at much greater distances than ordinarily.

Tyndall's "acoustic clouds" were regarded by him as being mainly due to the presence of an excess of aqueous vapour in some parts of the atmosphere. Now, moist air has a greater power of radiating heat than dry air, and the consequent "stifling" of the sound passing through very moist air† may be appreciable, although in air under ordinary conditions the effect is negligible. Tyndall does not give a definite idea of the nature of the action of an acoustic cloud, which may act, for the purposes of his theory, by stifling the sound, by scattering, or by reflecting it back to its starting point. Probably the first two effects both play their proper parts. Moreover, as we have shown, the backward reflection when sound enters such a dissipative medium tends to lose its periodicity, and to be independent of the reflecting medium when the dissipation exceeds a certain limit.

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* 'Theory of Sound,' 1896, § 290.

† Stokes, 'Phil. Mag.,' April, 1851; 'Math. and Phys. Papers,' vol. 3, p. 142.

The theory of Stokes shows that the k^2 of the ordinary equation of wave motion is to be replaced by $k^2\gamma(q+\iota n)/(q+\iota\gamma n)$, where $2\pi/n$ is the period of the sound, and q is the radiation constant.*

Since q is small compared with n, as proved by Stokes, and $\gamma = 1{\cdot}4$, it appears that $k_0\omega'$ is to be replaced by

$$k_0\omega'(1-\iota q/7n).$$

As a limiting case, we will suppose that q is zero in air, and that the variable medium of the preceding theory is the layer of transition between ordinary air and an acoustic cloud.

Assuming the transition to take place uniformly, we may replace q by one-half its final value. Thus, $k_0(\omega_2-\omega_1)$ is to be replaced by $k_0(\omega_2-\omega_1)(1-\iota q/14n)$. Now, the general effect of such a complex value of β has been already seen to be a tendency towards a certain "limiting" reflection of the sound.

The value of the quantity $\exp 2\iota k_0\beta$, when q is present, bears to the value, when q is zero, the approximate ratio $\exp \frac{1}{7}k_0(\omega_2-\omega_1)q/n$.

In considering the effect of q apart from that of the variation of wave-length, which is itself small, we may write $k_0(\omega_2-\omega_1) = k(x_2-x_1)$. The argument of the above exponential therefore becomes $q(x_2-x_1)/7\mathrm{V}$. If this have a value of about 5 only, there is a reflection which is almost limiting, since e^5 is about 144, a value sufficiently great in general for the above theory of limiting or non-periodic reflection to apply. In the absence of any evidence† that q is inappreciable for a cloud composed largely of vapour, it appears possible that an acoustic cloud may act in this manner, if the layer of transition have a length (x_2-x_1) sufficiently great.

In this case, all the phenomena noticed by Tyndall would seem capable of explanation by his theory. The order of effectiveness of sounds noticed on a day during which the atmosphere is uniform may change when acoustic clouds arise. For when they are present, the maxima and minima depending on the formula (12) may cease to be found.

Lord Rayleigh has pointed out that the presence of an acoustic fog would probably not influence a sound of very short duration, such as the report of

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* *Cf.* Lord Rayleigh, 'Theory of Sound,' vol. 2.

† *Vide* the Appendix.

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a gun, in the same manner as the prolonged sound given out by a siren. Moreover, the diffraction round obstacles may be less effective for sounds of short duration. These effects of the *duration* of the sound are, however, independent of the considerations treated at present.

Upward Propagation of Sound in the Atmosphere.

The presence in the atmosphere of a temperature gradient, and of a variation of density due to gravity, each cause the circumstances of propagation of sound to vary with the height. When sound travels upward, each will cause a certain fraction to be reflected back to the earth. We may consider the effects separately, since both are small, and will first treat that of the temperature gradient.

Let V be the velocity at a height z, and V_0 that at the earth. If (p, ρ), (p_0, ρ_0) are the corresponding pressures and densities, then in convective equilibrium

$$p\rho^{\gamma} = \text{constant},$$

and

$$V^2 = dp/d\rho = \gamma p_0 \rho^{\gamma-1}/\rho_0^{\gamma}.$$

Since

$$dp = -g\rho\, dz,$$

we deduce

$$V^2 = V_0^2 - (\gamma-1)gz. \tag{24}$$

If $2\pi/p$ is the period of the sound,

$$k_0 = p/V_0,$$

and

$$\omega' = (1-\epsilon z)^{-\frac{1}{2}}, \qquad \alpha = -\tfrac{1}{2}\epsilon\omega', \qquad \omega\epsilon = -2(1-\epsilon z)^{\frac{1}{2}},$$

where

$$\epsilon = (\gamma-1)g/V_0^2.$$

Hence, since $\omega' = 1$ at the surface, $z = 0$, the intensity ratio of the sound reflected from the stratum of height z, by the formula (12), becomes

$$R = \left\{1 + (1-\epsilon z)^{-1} - 2(1-\epsilon z)^{-\frac{1}{2}} \cos \frac{4p}{V_0\epsilon}\left(1-(1-\epsilon z)^{\frac{1}{2}}\right)\right\} \frac{\epsilon^2 V_0^2}{64p^2}.$$

Since ϵ is very small, we may write, retaining only its lowest powers,

$$R = \frac{(\gamma-1)^2 g^2}{16p^2 V_0^2}\left\{1 + \frac{(\gamma-1)gz}{2V_0^2}\right\} \sin^2 \frac{pz}{V_0}. \tag{25}$$

If λ_0 is the wave-length at the surface of the earth, there is extinction of reflected sound when

$$2\pi z/\lambda_0 = s\pi \quad \text{or} \quad z = \tfrac{1}{2}s\lambda_0, \tag{26}$$

where s is an integer.

For air under ordinary conditions we may write

$$\gamma = 1{\cdot}41, \qquad g = 32, \qquad V_0 = 1100 \text{ feet per second}, \quad \text{and} \quad p = 2\pi V_0/\lambda_0,$$

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whence we obtain

$$R = 13 \,.\, 10^{-15}\lambda_0^2\,\{1+21\,.\,10^{-7}z\}\,\sin^2 2\pi z/\lambda_0, \tag{27}$$

where z and λ_0 are measured in feet.

The total reflection due to the temperature gradient is thus very small, and is a periodic function of z whose amplitude increases very slowly with z in a linear manner. This discussion of the case of convective equilibrium gives a general idea of the order of magnitude and mode of variation of the effect for other distributions of temperature in the atmosphere.

Effect of the Gradient of Atmospheric Density on Upward Propagation.

With the usual notation, if z be measured upwards, the equation of upward propagation of sound in the atmosphere is

$$\rho\frac{\partial^2 u}{\partial t^2} = -\frac{\partial p}{\partial x},$$

where u is displacement, and, by Boyle's law,

$$p/\rho\tau = p_0\,(1+\partial u/\partial x)/\rho_0\tau_0.$$

Thus

$$\rho\frac{\partial^2 u}{\partial t^2} = V_0^2\frac{\partial}{\partial x}\left(\rho\frac{\partial u}{dx}\right). \tag{28}$$

If $2\pi/k_0$ is the wave-length at the surface of the earth, where the velocity is V_0, then

$$\frac{\partial}{\partial x}\left(\rho\frac{\partial u}{\partial x}\right)+k_0^2\rho u = 0,$$

or

$$\frac{\partial^2 u}{\partial y^2}+k_0^2\rho^2 u = 0, \tag{29}$$

where

$$y = \int^x dx/\rho. \tag{30}$$

The ω' of the formula (6) is now ρ and the ω is $\int^y \rho dy$ or x. Moreover,

$$\omega'' = \partial\rho/\partial y = \rho\,\partial\rho/\partial x = \rho\rho'.$$

Thus

$$\alpha = -\omega''/\omega'^2 = -\rho'/\rho.$$

But

$$\log \rho_0/\rho = gx/k,$$

where k is the compressibility.

Thus

$$\alpha_1 = \alpha_2\,g/k,$$

and therefore by the formula (12) the intensity ratio of sound reflected from a height x is

$$R = g^2 \sin^2 .\, k_0\beta/4k_0^2k^2. \tag{31}$$

For air under ordinary conditions, $g = 32$, $k = (1100)^2/1{\cdot}41$.* Performing the calculation, it appears that for sound whose wave-length, expressed in feet, is λ,

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* g/k is so small that this result holds even for very small heights, examining ψ'' of (4)).

$$R = 8{\cdot}7 \,.\, 10^{-12} \,.\, \lambda^2 \sin^2 2\pi z/\lambda. \tag{32}$$

This effect is of the same character as that of the temperature gradient. For a given height, the effect of the variation of density is about 700 times as great as that of the temperature variation, provided that the ratio of height to wave-length is great.

The type of analysis here employed is applicable to many other physical problems connected with short waves, such, for example, as that of short waves of transverse displacement in an infinite stretched string.

APPENDIX.—*On Viscosity, Radiation, and Conduction, and their Effect on the Propagation of Sound in a Variable Medium.*

This appendix is an amplification of the previous discussion of stifling by moist air. It was there tacitly assumed that dissipation of sound took place merely by radiation of heat during its passage. This was based on the fact (*infra*) that radiation is the factor most liable to cause appreciable dissipation in sounds of any pitch whatever.

Other factors capable of causing it for a smaller range of pitch are viscosity and conduction.

If μ is the kinematic viscosity, and ν the thermometric conductivity, then the equation of propagation of sound, when these dissipative agencies alone act, is*

$$u = e^{-mx} \,.\, \cos(nt - x/V),$$

where

$$m = \frac{n^2}{2V^3}\left(\frac{4}{3}\mu + \frac{\gamma-1}{\gamma}\,.\,\nu\right).$$

In C.G.S. units $\mu = 1{\cdot}4$, $\nu = 0{\cdot}26$,†

and thus $m = 0{\cdot}12\, n^2/V^3$.

Now V = 33,200 cm. per second, and for a sound of frequency $n/2\pi = 256$, $m = 1{\cdot}02 \,.\, 10^{-8}$ cm.$^{-1}$.

It was shown above that the quantity $\exp 2\iota k_0\beta$ of reflected sound in a variable medium, when radiation is present, bears to that when radiation is absent the ratio

$$\exp \tfrac{1}{7}q\,(x_2 - x_1)/V, \quad \text{or} \quad \exp m_1 (x_2 - x_1).$$

Here m_1 is the quantity in the dissipation ratio $e^{-m_1 x}$ due to radiation alone, which occurs in ordinary propagation in a uniform medium.

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* Kirchhoff, 'Pogg. Ann.,' vol. 134, p. 177, 1868; Lord Rayleigh, 'Theory of Sound,' 2nd edition, § 348.

† E. Müller, 'Wied. Ann.,' vol. 60, p. 113, 1896.

Thus $$m_1 = q/7\mathrm{V} = 0{\cdot}44 \,.\, 10^{-5} q \text{ cm.}^{-1},$$
and $$m_1/m = 436q.$$

Thus if q be not less than about 1/500, radiation is the predominant factor in dissipation of sounds of low pitch, and, therefore, also in causing limiting reflection when the medium is not uniform.

For a sound of very high pitch, radiation is not so necessarily predominant. In the case of a sound of frequency 5000, $m_1/m = q$ (approx.).

As to the value of q, much divergence of opinion exists. Wilmer Duff,[*] in a series of experiments on the rate of fall of intensity of sounds of frequency about 7000, concluded that the great rate he obtained could only be explained by assigning to q a value 8·3 in ordinary air. Apparently valid reasons were given for supposing that atmospheric refraction and internal reflection had not played any appreciable part in affecting the rate of fall, for the conditions of the atmosphere and the circumstances of position appear to have varied widely. Moreover, wind was almost absent, and the vertical temperature gradient, the other chief cause of refraction, must have been subject to variation, as observations were taken both at noon and after sunset.

But Lord Rayleigh,[†] in examining these results, concluded that the radiating power must be hundreds of times smaller, and attributed Duff's results to atmospheric refraction, or to some hitherto neglected cause. He suggested, as a possible cause, a delay in the equalisation of different kinds of energy in a compressed gas not insensible in comparison with a period of sound, and thus causing dissipation. This criticism was supported by an argument based on the theory of exchanges, and by an experiment on the lines of that used by Clement and Desormes in determining γ. The agreement of this experiment with corresponding theory[‡] appears to be conclusive, and we must therefore suppose that q is not of order unity for ordinary air. This justifies the assumption made in a previous portion of this paper that $q = 0$ for ordinary air. But for very moist air the value of q does not appear to have been examined experimentally, and we have no indirect knowledge of it, except that it is greater than in ordinary air. That its value may be considerable in comparison with that for dry air seems fairly certain. Moreover, the viscosity μ and conductivity ν may be, and probably are, considerably greater in moist air, thus lending support to the argument based on (18), by adding their effects to those of q (which latter probably predominates, except for sounds of very high pitch).

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[*] 'Physical Review,' March, 1898.

[†] 'Phil. Mag.,' vol. 47, p. 308, 1899.

[‡] 'Phil. Mag.,' vol. 47, p. 314, 1899.

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Until the values of these quantities (μ, ν, q), and more especially of q, are known for very moist air, the explanation given by Tyndall of his own observations must be regarded as a very possible one, for the change of character of the reflection effect of (12) may readily occur in a sufficiently large layer, if the three coefficients (μ, ν, q) be at all appreciably modified by the presence of moisture. We may note that the effect of q alone is independent of the pitch.

On the Nature of the Streamers in the Electric Spark.

By S. R. Milner, D.Sc. (Lond.), Lecturer in Physics, the University of Sheffield.

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(Abstract.)

The main subject of the work described in the present paper consists in the examination of the streamers in the inductive spark in the monochromatic lights of the various metallic lines. It thus forms an extension of the research of Messrs. Schuster and Hemsalech,* in which the examination of the streamers was restricted to the inductionless spark. The observations were taken by photographing the spectrum as drawn out by a rotating mirror, the slit of the spectroscope being removed and replaced by the spark itself, so that each line of the spectrum formed a monochromatic image of the spark. In order to avoid the superposition of the series of streamers which are formed in the drawing out of each monochromatic image, an arrangement of the prisms of the spectroscope was used by which, while the images of the spark on the camera screen were vertical, and drawn out in a horizontal direction, the dispersion of the spectrum was in a direction of 45° to the horizontal. By this arrangement the series of streamers corresponding to each metallic line becomes distinctly separated from the others.

Photographs of the streamers in the spectra of the sparks from the following metallic poles were taken, in each case with a number of different inductances in series with the spark: aluminium, antimony, bismuth, cadmium, calcium, copper, lead, magnesium, mercury, nickel, platinum,

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* 'Phil. Trans.,' A, vol. 193, p. 189 (1900).